General OPF Problems with Reactive Power Costs: A Distributed SDP Approach

Chin-Yao Chang and Wei Zhang

arXiv:1612.04508v1 [math.OC] 14 Dec 2016

Abstract—**We consider a general class of the optimal power flow (OPF) problems with additional reactive power costs and apparent power constraints as opposed to the standard formulation. The reactive power objective functions are shown in various OPF formulations for voltage support and microgrid applications. The apparent power constraints are introduced by the inverter-interfaced renewable energy resources (RERs) or energy storage system (ESSs). Such OPF problem is challenging to solve due to its non-convexity. Recent studies show that semidefinite programming (SDP) approaches can either provide an exact or near global optima for many OPF problems. However, those existing algorithms can not be directly applied to the OPF problems that we consider. In this paper, we rewrite the OPF problem as a combination of several non-convex subproblems. Analysis on those subproblems leads to conditions of exact SDP convex relaxation. Furthermore, with perturbations on the distributed OPF formulation, we develop algorithms to find a near global optimum for the general OPF problem. A bound on the difference between the near global solution and the optimum is also found. All the results hold true for cases with quadratic reactive power cost and apparent power constraints of inverter-interfaced resources. The proposed algorithm is validated through numerical examples.**

Index Terms—**Optimal power flow, semidefinite programming convex relaxation, distributed optimization**

I. Introduction

Electrical power grid is under a major change with high penetration of RERs. The transition increases the need of accurate and real time OPF solutions. However, the OPF problems are challenging to solve in general because they are large-scale and non-convex optimization problems. Various algorithms have been developed for the OPF problem, including Newton-Raphson [1], interior point methods [2], quadratic programming [3], nonlinear programming [4], particle swarm optimization [5], [6], to name a few. Many algorithms mentioned above only guaranteed a local optimum due to the nonconvexity of the OPF problem. Recent works show that SDP can efficiently find a global optimal solution of the OPF problem for many power networks [7], [8]. However, those algorithms still limited to a certain class of OPF problem and inevitably require load over-satisfaction assumption for general cases. Our interest is developing new SDP algorithms to efficiently find exact or near global optimal solutions of general OPF problems.

Conditions of exact SDP convex relaxation vary between radial (tree) and mesh network topologies. Tree network topology has a one-to-one correspondence between line power flows and nodal power injections. The fact implies that in tree networks, the power flow equations (PFEs) of all buses combine to be a bijective function of nodal apparent power and the phasor voltage with a given reference angle. The non-convex constraint sets of the OPF problem are easier to be characterized with the property. The SDP convex relaxation on tree topology is then more well-established. The work [9] shows that the OPF problem can be solved by SDP if the objective function is strictly increasing with respect to the voltage and the lower bounds of the reactive power are sufficiently small. Second-order cone programming (SOCP) convex relaxation is also developed for tree networks with general cost functions [10], while it requires existence of an infinite bus and other mild assumptions on line power flows. Convex relaxations by SOCP and SDP are equivalent for tree networks in the sense that they have the same feasible region [11]. However, SDP relaxation has a tighter feasible region compared to SOCP for mesh networks [11].

Conditions of exact SDP convex relaxation for mesh networks are relatively less understood due to a more non-trivial feasible region. Some test cases also verify that SDP may not provide a feasible solution for mesh networks [12]. The SDP-based approaches inevitably require load over-satisfaction assumptions to yield a physical viable solution as found in [7], [13]. Many works then instead focus on finding a near global optimum [8], [14]. Those algorithms are shown successful in many test cases. However, they still have some limitations such as tuning on the reactive power cost functions [8], or convergence issues [14].

The aforementioned SDP-based approaches focus on OPF problems with objective functions of active power and line power losses. The result can not be directly applied to some OPF formulations with reactive power objective functions. For example, reactive power market for voltage support [15] and reactive power sharing between inverters in microgrids [16]. The existing SDP methods also have not considered constraints on apparent power that are shown in inverter-interfaced energy resources [17], [18].

In this paper, we extend the existing SDP results to solve the OPF problem with reactive power cost and apparent power constraints. We start with the observation that every constraint of the OPF problem is only related to the voltage of one bus and its neighboring buses. Such property naturally leads to an equivalent form of the OPF problem, which is written as a combination of several non-convex subproblems. Each subproblem has the decision variable as the voltage of one bus and its neighbors. Those subproblems are correlated by equality constraints on the copies of the voltage of connected buses. We propose to convexify the distributed formulation of the OPF problem instead of the direct SDP convex relaxation on the original OPF problem as found in the literature.

C.-Y. Chang is with the Department of Mechanical and Aerospace Engineering, University of California, San Diego, W. Zhang is with the Department of Electrical and Computer Engineering, Ohio State University. Email: chc433@ucsd.edu, zhang@ece.osu.edu

The novel convexification method leads to new conditions of exact convex relaxation of the OPF problem which involves quadratic cost on the reactive power and constraints on the apparent power commonly seen in RERs and ESSs. Furthermore, algorithms of finding a near global optimum are developed based on the distributed formulation of the OPF problem.

We propose different algorithms for near global optima based on the topology of the electrical networks. For tree networks, a global optimum can always be found by SDP when the cost function is strictly increasing and the lower bounds of the reactive power constraints are sufficiently small [9]. We extend the results to find a near global optimum for OPF problems with convex quadratic cost functions on the active and reactive power. The extension is accomplished by adding small perturbations on the distributed formulation of the OPF problem. For general mesh networks, we further manipulate the perturbed OPF problem and develop an algorithm for a near global optimum. Remarkably, we can show that the error is bounded by the angle difference between the connected buses, which is typically small. Although the derived near global optimum may violate power constraints at some buses, the bounded violation is considered acceptable due to the predictive nature of those constraints.

The rest of the paper is organized as follows. We formulate the OPF problem in Section II. In Section III, we propose a distributed formulation of the OPF problem derive conditions of exact SDP convex relaxation. We further develop near global optimal algorithms in Section IV. The proposed algorithms are validated in Section V. We conclude the paper in Section VI.

Notation The trace of a matrix A is written as $\mathrm{Tr}\{A\}$. The cardinality of a set $\mathcal{N}$ is denoted as $|\mathcal{N}|$. Let $\mathbb{R}_+$ and $\mathbb{S}_+$ be the sets of non-negative real number and positive semidefinite matrix, respectively. For a set of matrix W_i, $i \in \mathcal{I}$, let $W_\mathcal{I}$ be the collection of matrices of W_i $\forall i \in \mathcal{I}$. Denote $[N]$ as a set of integer $\{1, 2, ..., N\}$. For a complex number $a \in \mathbb{C}$, let $|a|$ and $\angle a$ be the complex modulus and angle, respectively. For a complex matrix A, let A^* be the conjugate transpose.

II. Problem Setup

Consider an electrical networks with a set of buses, generation, and load buses given as $\mathcal{N}$, $\mathcal{N}_g$ and $\mathcal{N}_L$. We denote the set of inverter-interfaced resources as $\mathcal{N}_I \subseteq \mathcal{N}_g$. Denote N as the cardinality of $\mathcal{N}$. Let $V = Ee^{j\theta} \in \mathbb{C}^N$ be the collection of phasor voltage of all buses, where $E \in \mathbb{R}^N$ and $\theta \in [0, 2\pi)^N$ are the voltage magnitude and angle, respectively. The PFEs is given as [19]

$$P_k = \mathrm{Tr}\{Y_k VV^*\} + P_{D_k}, \qquad Q_k = \mathrm{Tr}\{\bar{Y}_k VV^*\} + Q_{D_k},$$

where P_{D_k} and Q_{D_k} are the active and reactive power demands at bus k, and Y_k, $\bar{Y}_k$ are derived from the admittance matrix $\mathbf{Y}$, see [19] for details.

For secure operation of electrical networks, the voltage and power injections of buses should always satisfy the following constraints

$$\underline{V}_k^2 \le |V_k|^2 \le \bar{V}_k^2, \ \forall k \in \mathcal{N} \tag{1a}$$

$$\underline{P}_k \le P_k \le \bar{P}_k, \ \ \underline{Q}_k \le Q_k \le \bar{Q}_k, \ \forall k \in \mathcal{N} \tag{1b}$$

$$P_k^2 + Q_k^2 \le \bar{S}_k^2, \forall k \in \mathcal{N}_I \tag{1c}$$

$$|V_i - V_k|^2 \le \bar{V}_{ik}, \forall \{i, k\} \in \mathcal{E}, \tag{1d}$$

where V_k is the phasor voltage at bus k, $\bar{V}_{ik}$ is the bound of the voltage difference between connected buses, $\bar{S}_k$ is the apparent power bound for the inverter buses, $\underline{V}_k$ and $\bar{V}_k$ are the lower and upper bounds of the voltage magnitude respectively. All $\underline{P}_k, \underline{Q}_k, \bar{P}_k, \bar{Q}_k$ are defined similarly. Note that the load buses are typically characterized by constant power loads. In such case, $\underline{P}_k = \underline{Q}_k = \bar{P}_k = \bar{Q}_k = 0$ for $k \in \mathcal{N}_L$. For $k \in \mathcal{N}_I$, we impose constraints (1b) along with the apparent power constraints (1c) to account the constraints of the power factor of the inverters in some applications [18].

The cost function of the OPF problem is typically given as a combination of generation cost and power losses

$$\sum_{k\in\mathcal{N}_g} \Big(c_{k2}P_k^2 + c_{k1}P_k + d_{k2}Q_k^2 + d_{k1}Q_k\Big) \tag{2}$$
$$+ \sum_{\{i,k\}\in\mathcal{E}} w_{0,ik}\Big(P_{ik}^{loss} + Q_{ik}^{loss}\Big),$$

where $w_{0,ik} \in \mathbb{R}_+$ is the weighting factor of the line power losses, $c_{k2}, d_{k2} \in \mathbb{R}_+$, and $c_{k1}, d_{k1} \in \mathbb{R}$. The resistive power dissipation P_{ik}^{loss} and reactive power stored in the inductance Q_{ik}^{loss} of every edge $\{i, k\} \in \mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ are given as

$$P_{ik}^{loss} + jQ_{ik}^{loss} := \mathbf{Y}_{ik}\big(|V_i|^2 + |V_k|^2 - V_iV_k^* - V_kV_i^*\big).$$

Define the decision variable as $W = VV^*$. The overall OPF problem is written as follows:

$$\textbf{(P1)} \quad \min_{a,b,W} \sum_{k\in\mathcal{N}_g} (a_k + b_k) + \sum_{\{i,k\}\in\mathcal{E}} w_{ik}\mathrm{Tr}\{M_{ik}W\},$$

subject to

$$\begin{bmatrix} c_{k1}(\mathrm{Tr}\{Y_kW\} + P_{D_k}) - a_k & \star \\ \sqrt{c_{k2}}(\mathrm{Tr}\{Y_kW\} + P_{D_k}) & -1 \end{bmatrix} \preceq 0, \ \forall k \in \mathcal{N}_g, \tag{3a}$$

$$\begin{bmatrix} d_{k1}(\mathrm{Tr}\{\bar{Y}_kW\} + Q_{D_k}) - b_k & \star \\ \sqrt{d_{k2}}(\mathrm{Tr}\{\bar{Y}_kW\} + Q_{D_k}) & -1 \end{bmatrix} \preceq 0, \tag{3b}$$

$$\begin{bmatrix} -\bar{S}_k^2 & \star & \star \\ \mathrm{Tr}\{Y_kW\} + P_{D_k} & -1 & \star \\ \mathrm{Tr}\{\bar{Y}_kW\} + Q_{D_k} & 0 & -1 \end{bmatrix} \preceq 0, \ \forall k \in \mathcal{N}_I \tag{3c}$$

$$\underline{P}_k \le \mathrm{Tr}\{Y_kW\} + P_{D_k} \le \bar{P}_k, \ \forall k \in \mathcal{N} \tag{3d}$$

$$\underline{Q}_k \le \mathrm{Tr}\{\bar{Y}_kW\} + Q_{D_k} \le \bar{Q}_k, \tag{3e}$$

$$\underline{V}_k^2 \le \mathrm{Tr}\{M_kW\} \le \bar{V}_k^2, \tag{3f}$$

$$\mathrm{Tr}\{M_{ik}W\} \le \bar{V}_{ik}, \ \forall\{i, k\} \in \mathcal{E}, \tag{3g}$$

$$W \succeq 0 \quad \mathrm{rank}(W) = 1, \tag{3h}$$

where $\star$ denotes the complex conjugate of the off-diagonal elements of the Hermitian matrices, $w_{ik} = w_{0,ik}(\mathrm{Re}\{\mathbf{Y}_{ik}\} + \mathrm{Im}\{\mathbf{Y}_{ik}\})$, M_k and M_{ij} are defined so that $\mathrm{Tr}\{M_kW\} = |V_k|^2$ and $\mathrm{Tr}\{M_{ik}W\} = |V_i - V_k|^2$. By the introduction of a, b and Schur complement, the quadratic cost (2) is encoded in the constraints (3a) and (3b). Schur complement is also used to replace the quadratic terms of the apparent power constraints of $\mathcal{N}_I$ to constraint (3c). The combined constraints $W \succeq 0$ and $\mathrm{rank}(W) = 1$ correspond to writing the voltage as a matrix variable.

III. Distributed Optimization Approach

Solving **(P1)** is NP-hard in general [20]. In this section, we will first develop an equivalent distributed optimization problem to **(P1)**. The non-convex constraints of the subproblems are dropped. Conditions of finding a global optimal solution of **(P1)** from the convex relaxation are then derived.

A. Distributed Optimization Formulation

All the constraints in **(P1)** either relate to power injections at one bus or voltage difference of connected buses. In addition, the power injection at one bus only related to the voltage of the connected buses due to the nature of PFEs. Let $\hat{V}_k$ be a collection of voltage of bus k and the nodes connected to bus k. Constraints (3a)-(3g) can be rewritten by replacing W with $W_k := \hat{V}_k \hat{V}_k^*$

$$\begin{bmatrix} c_{k1}(\text{Tr}\{Y_{k,r}W_k\} + P_{D_k}) - a_k & \star \\ \sqrt{c_{k2}}(\text{Tr}\{Y_{k,r}W_k\} + P_{D_k}) & -1 \end{bmatrix} \preceq 0,\ \forall k \in \mathcal{N}_g \tag{4a}$$

$$\begin{bmatrix} d_{k1}(\text{Tr}\{\bar{Y}_{k,r}W_k\} + Q_{D_k}) - b_k & \star \\ \sqrt{d_{k2}}(\text{Tr}\{\bar{Y}_{k,r}W_k\} + Q_{D_k}) & -1 \end{bmatrix} \preceq 0, \tag{4b}$$

$$\begin{bmatrix} -\bar{S}_k^2 & \star & \star \\ \text{Tr}\{Y_{k,r}W_k\} + P_{D_k} & -1 & \star \\ \text{Tr}\{\bar{Y}_{k,r}W_k\} + Q_{D_k} & 0 & -1 \end{bmatrix} \preceq 0,\ \forall k \in \mathcal{N}_I \tag{4c}$$

$$\underline{P}_k \le \text{Tr}\{Y_{k,r}W_k\} + P_{D_k} \le \bar{P}_k,\ \forall k \in \mathcal{N}, \tag{4d}$$

$$\underline{Q}_k \le \text{Tr}\{\bar{Y}_{k,r}W_k\} + Q_{D_k} \le \bar{Q}_k, \tag{4e}$$

$$\underline{V}_k^2 \le \text{Tr}\{M_{k,r}W_k\} \le \bar{V}_k^2, \tag{4f}$$

$$\text{Tr}\{M_{ik,r}W_k\} \le \bar{V}_{ik},\ \forall \{i,k\} \in \mathcal{E}, \tag{4g}$$

where $M_{k,r}$ is a principal submatrix of M_k by dropping the rows and columns associated with buses $\mathcal{N} \setminus \mathcal{N}_k$, $\mathcal{N}_k := \{i | \{i,k\} \in \mathcal{E}\} \cup \{k\}$, $M_{ik,r}$, $Y_{k,r}$, $\bar{Y}_{k,r}$ are defined similarly. Instead of solving the **(P1)**, we view $W_{\mathcal{N}}$ as a new state variable and consider the following distributed convex optimization problem

$$\textbf{(P2)}\ \min_{a,b,W_{\mathcal{N}}} \sum_{k\in\mathcal{N}_g} (a_k + b_k) + \sum_{\{i,k\}\in\mathcal{E}} w_{ik}\text{Tr}\{M_{ik}W_k\},$$

subject to

$$\begin{aligned} &\text{Eq. (4) holds,} \quad W_k \succeq 0, \quad \forall k \in \mathcal{N} \\ &W_k(\hat{k},\hat{k}) = W_i(\hat{k},\hat{k}),\ \ \forall \{i,k\} \in \mathcal{E}, \\ &W_k(\hat{i},\hat{k}) = W_i(\hat{i},\hat{k}), \end{aligned}$$

where $\hat{k}$ denotes the row (or column) of W_i associated with bus k. We rewrite the equality constraints of W_i and W_k, $\{i,k\} \in \mathcal{E}$, in the following matrix form for convenience

$$\begin{aligned} &B_{1,ki}W_k = B_{2,ik}W_i, \\ &B_{l,ki}W_k = B_{l,ik}W_i, l = 3,4, \end{aligned} \tag{5}$$

where

$$B_{1,ki}(l,m) = \begin{cases} 1, \text{ if } l = m = \hat{k}, \\ 0, \text{ otherwise} \end{cases}$$

$$B_{2,ik}(l,m) = \begin{cases} 1, \text{ if } l = m = \hat{k}, \\ 0, \text{ otherwise} \end{cases}$$

$$B_{3,ki}(l,m) = \begin{cases} 1, \text{ if } (l,m) = (\hat{k},\hat{i}) \text{ or } (l,m) = (\hat{i},\hat{k}), \\ 0, \text{ otherwise} \end{cases}$$

$$B_{4,ki}(l,m) = \begin{cases} -j, \text{ if } (l,m) = (\hat{k},\hat{i}), \\ j, \text{ if } (l,m) = (\hat{i},\hat{k}), \\ 0, \text{ otherwise} \end{cases}$$

Notice that **(P2)** can be viewed as the coupled optimization of N subproblems. Each subproblem $k \in \mathcal{N}$ is a relaxed OPF associated with a star network centered at node k. The subproblems are called "relaxed" because no voltage nor power injection constraints are imposed on all the leaf nodes other than the equality constraints that relate the neighboring subproblems. With the equality constraints (5), the optimal solution of **(P1)** can be reconstructed if rank$(W_k^{opt}) = 1$ for all $k \in \mathcal{N}$. Define $\hat{V}_k^{opt} \in \mathbb{C}^{N_k}$, $N_k := |\mathcal{N}_k|$, such that $W_k^{opt} = \hat{V}_k^{opt}(\hat{V}_k^{opt})^*$. The optimal solution of **(P1)** can be derived from $\hat{V}_k^{opt}$, $k \in \mathcal{N}$.

Proposition 1. *If the optimal solution of* ***(P2)*** *has* rank$(W_k) = 1$, $\forall k \in \mathcal{N}$, *then there exists* $\theta_{r,k} \in [0, 2\pi)^n$ *such that* $V^{opt} \in \mathbb{C}^n$ *with* $V^{opt}(k) := \exp^{j\theta_{r,k}} \hat{V}_k^{opt}(\hat{k})$ *is the optimal solution of* ***(P1)****.*

Proposition 1 follows immediately if a reference angle is chosen and $\theta_{r,k} \triangleq \theta_r, \forall k \in \mathcal{N}$.

B. Exact Convex Relaxation

A common assumption on the OPF problem **(P1)** is the strict feasibility. We will adopt the assumption in the rest of the paper. Under the assumption, Slater condition holds for **(P2)** and the strong duality follows for the convex optimization **(P2)**. The strong duality can lead to a low rank solution of W_k for **(P2)**. Define $y = [\bar{\lambda}, \underline{\lambda}, \bar{\gamma}, \underline{\gamma}, \bar{\mu}, \underline{\mu}, \bar{\zeta}] \ge 0$ as a collection of the dual variables associated with all the constraints in **(P2)** except constraints (4a-4c) and (5), where $\bar{\lambda}, \underline{\lambda}, \bar{\gamma}, \underline{\gamma}, \underline{\mu}, \bar{\mu} \in \mathbb{R}_+^N$ and $\bar{\zeta} \in \mathbb{R}_+^{|\mathcal{E}|}$. Let $h = [h_{1,ik}, h_{2,ik}, h_{3,ik}, h_{4,ik}]_{\{i,k\}\in\mathcal{E}}$, $h_{l,ik} = -h_{l,ki}$, be the collection of the dual variables associated with constraints (5). Define $R_k^p, R_k^q \in \mathbb{S}_+^2, R_k^r \in \mathbb{S}_+^3$ as the dual variables associated with the matrix inequalities (4a-4c). We write $R = \{R_{\mathcal{N}_g}^p, R_{\mathcal{N}_g}^q, R_{\mathcal{N}_I}^r\}$ for convenience. The dual of **(P2)** is given as follows

$$\textbf{(DP2)} \qquad \max_{y\ge 0,h,R} \Big(\min_{W_{\mathcal{N}}\succeq 0} L(y,h,R,W_{\mathcal{N}}) \Big),$$

where

$$R_k^p = \begin{bmatrix} 1 & \star \\ r_{k,12}^p & r_{k,22}^p \end{bmatrix} \succeq 0, R_k^q = \begin{bmatrix} 1 & \star \\ r_{k,12}^q & r_{k,22}^q \end{bmatrix} \succeq 0,$$

$$R_k^r = \begin{bmatrix} 1 & \star & \star \\ r_{k,12}^r & r_{k,22}^r & \star \\ r_{k,13}^r & r_{k,23}^r & r_{k,33}^r \end{bmatrix} \succeq 0$$

$$\begin{aligned} L(\cdot) = &\sum_{k\in\mathcal{N}} \Big(\underline{\lambda}_k(\underline{P}_k - P_{D_k}) + \bar{\lambda}_k(-\bar{P}_k + P_{D_k}) \\ &+ \underline{\gamma}_k(\underline{Q}_k - Q_{D_k}) + \bar{\gamma}_k(-\bar{Q}_k + Q_{D_k}) + \underline{\mu}_k \underline{V}_k^2 \\ &- \bar{\mu}_k \bar{V}_k^2 + \text{Tr}\{A_k(\cdot)W_k\} \Big) - \sum_{\{i,k\}\in\mathcal{E}} \bar{\zeta}_{ik}\bar{V}_{ik} \\ &+ \sum_{k\in\mathcal{N}_g} \Big((2r_{k,12}^p\sqrt{c_{k2}} + c_{k1})P_{D_k} - r_{k,22}^p \\ &+ (2r_{k,12}^q\sqrt{d_{k2}} + d_{k1})Q_{D_k} - r_{k,22}^q \Big) \end{aligned}$$

$$+\sum_{k\in\mathcal{N}_I}\Big(2r^r_{k,12}P_{D_k}+2r^r_{k,13}Q_{D_k}-r^r_{k,22}-r^r_{k,33}\Big),$$

$$A_k(\cdot)=\sum_{\{i,k\}\in\mathcal{E}}\left(\sum_{m=[4]}h_{m,ki}B_{m,ki}+(\bar{\zeta}_{ik}+w_{ik})M_{ik,r}\right)$$
$$+(\bar{\mu}_k-\underline{\mu}_k)M_{k,r}+\hat{\lambda}_kY_{k,r}+\hat{\gamma}_k\bar{Y}_{k,r}$$

$$\hat{\lambda}_k=\begin{cases}\bar{\lambda}_k-\underline{\lambda}_k+c_{k1}+2r^p_{k,12}\sqrt{c_{k2}}, & k\in\mathcal{N}_g\setminus\mathcal{N}_I\\ \bar{\lambda}_k-\underline{\lambda}_k+2r^r_{k,12}+c_{k1}+2r^p_{k,12}\sqrt{c_{k2}}, & k\in\mathcal{N}_I\\ \bar{\lambda}_k-\underline{\lambda}_k, & \text{otherwise}\end{cases}$$

$$\hat{\gamma}_k=\begin{cases}\bar{\gamma}_k-\underline{\gamma}_k+d_{k1}+2r^q_{k,12}\sqrt{d_{k2}}, & k\in\mathcal{N}_g\setminus\mathcal{N}_I\\ \bar{\gamma}_k-\underline{\gamma}_k+2r^r_{k,13}+d_{k1}+2r^q_{k,12}\sqrt{d_{k2}}, & k\in\mathcal{N}_I\\ \bar{\gamma}_k-\underline{\gamma}_k, & \text{otherwise}\end{cases}$$

It can be shown that $A_k(\cdot)\succeq 0$ for all $k\in\mathcal{N}$. Otherwise,

$$\max_{y\geq 0,h,R\succeq 0}\Big(\min_{W_{\mathcal{N}}\succeq 0}L(\cdot)\Big)\leq\min_{W_{\mathcal{N}}\succeq 0}\Big(\max_{y\geq 0,h,R\succeq 0}L(\cdot)\Big)=-\infty,$$

which contradicts the fact that the optimal solution of **(P2)** is always bounded from below (recall the strong duality). Since $A_k(\cdot)\succeq 0$, the optimal W^{opt}_k should satisfy $\mathrm{Tr}\{A^{opt}_kW^{opt}_k\}=0$ for all $k\in\mathcal{N}$. The optimization **(DP2)** is then compactly rewritten as

$$\textbf{(DP2)}\qquad \max_{y\geq 0,h,R}L_0(y,h,R),\tag{6}$$

subject to

$$A_k(y,h,R)\succeq 0,\quad \forall k\in\mathcal{N},$$
$$R^p_k\succeq 0,\quad R^q_k\succeq 0,\quad \forall k\in\mathcal{N}_g\setminus\mathcal{N}_I,$$
$$R^r_k\succeq 0,\quad \forall k\in\mathcal{N}_I,$$

where $L_0(\cdot)$ is reduced from $L(\cdot)$ by dropping $\mathrm{Tr}\{A_kW_k\}$. Let A^{opt}_k denotes $A_k(y^{opt},h^{opt},R^{opt})$ for convenience. We have $\mathrm{rank}(W^{opt}_k)=1$ if $\mathrm{rank}(A^{opt}_k)=N_k-1$ because $\mathrm{Tr}\{A^{opt}_kW^{opt}_k\}=0$. We state the conditions of $\mathrm{rank}(A^{opt}_k)\geq N_k-1$ for $\mathrm{rank}(W^{opt}_k)\leq 1$ in Proposition 2.

Proposition 2. *If the i^{th} diagonal element of A^{opt}_k: $A^{opt}_{k,ii}>0$ for all $i\in[N_k]\setminus\{\hat{k}\}$, then $\mathrm{rank}(A^{opt}_k)=N_k-1$ and $\mathrm{rank}(W^{opt}_k)=1$.*

Proof. A_k is a linear combination of $M_{ik,r}$, $Y_{k,r}$, $\bar{Y}_{k,r}$, $M_{k,r}$, and $B_{m,ik,r}, m\in[4]$. The non-zero elements of those sparse matrices coincide, resulting in the possible non-zero entries of A_k shown in Eq. (7), where $\hat{k}:=1$. In the rest of this paper, we will have $\hat{k}=1$ for W_k or A_k without loss of generality. If $A^{opt}_{k,ii}>0$ for all $i=2,...,N_k$, then every column vector, except the column vectors associated with $\hat{k}$, are independent of each other. Accordingly, $\mathrm{rank}(A^{opt}_k)\geq N_k-1$. If $\mathrm{rank}(A^{opt}_k)=N_k$, then $W_k=0$, which contradicts with the voltage magnitude constraints $\mathrm{Tr}\{M_{k,r}W_k\}\geq \underline{V}^2>0$. Therefore, $\mathrm{rank}(A^{opt}_k)$ can only be N_k-1 and $\mathrm{rank}(W_k)=1$.

$$A_k=\begin{bmatrix}A_{k,11} & A_{k,12} & \cdots & A_{k,1N_k}\\ A_{k,21} & A_{k,22} & 0 & 0\\ \vdots & 0 & \ddots & \vdots\\ A_{k,N_k1} & 0 & \ldots & A_{k,N_kN_k}\end{bmatrix}\tag{7}$$

□

There are no necessary and sufficient conditions to ensure a matrix being positive semidefinite in general. However, since the underlying graph of A_k is a star graph, we can derive useful necessary and sufficient conditions for $A_k\succeq 0$ as follows

Proposition 3. *A matrix associated with a star network, or A_k shown in Eq. (7), is positive semidefinite if and only if the following conditions hold*

1) *$A_{k,ii}\geq 0$ for all $i\in[N_k]$,*
2) *$A_{k,ij}=0$ if $A_{k,jj}=0$ for $j>1$,*
3) *$A_{k,11}\geq\sum_{\{j\neq 1|A_{k,jj}>0\}}\frac{|A_{k,ij}|^2}{A_{k,jj}}$.*

Proof. Define $\bar{A}_k$ as a principal submatrix of A_k where the rows and columns with $A_{k,jj}=0$ are dropped. Let c be the cardinality of the set $\{j\neq 1|A_{k,jj}>0\}$. By using the Schur complement for c times, we have $\bar{A}_k\succeq 0$ if and only if $A_{k,11}\geq\sum_{\{j\neq 1|A_{k,jj}>0\}}\frac{|A_{k,ij}|^2}{A_{k,jj}}$, which completes the proof. □

We introduce our first main result in Theorem 1

Theorem 1. *If **(DP2)** is strictly feasible and the following conditions hold*

a) *Load over-satisfaction: $\underline{P}_k=\underline{Q}_k=-\infty$, $\forall k\in\mathcal{N}$,*
b) *Generation cost for $k\in\mathcal{N}_g\setminus\mathcal{N}_I$:*

$$2c_{k2}P_k+c_{k1}\geq 0,\ \textit{for all}\ P_k\in[\underline{P}_k,\bar{P}_k],$$
$$2d_{k2}Q_k+d_{k1}\geq 0,\ \textit{for all}\ Q_k\in[\underline{Q}_k,\bar{Q}_k],$$

c) *Generation cost for $k\in\mathcal{N}_I$:*

$$2(c_{k2}+p_{k1})P_k+c_{k1}\geq 0,\forall p_{k1},\in[0,1],$$
$$2(d_{k2}+q_{k1})Q_k+d_{k1}\geq 0,\forall q_{k1}\in[0,1],$$
$$\forall P_k,Q_k\ \textit{s.t. Eq. (4c)-(4e) hold},$$

*then the optimal solution of **(P2)** has $\mathrm{rank}(W^{opt}_k)=1,\forall k\in\mathcal{N}$, and there exists $\theta_r\in[0,2\pi)^n$ such that $V^{opt}\in\mathbb{C}^n$ with $V^{opt}(k):=\exp^{j\theta_{r,k}}\hat{V}^{opt}_k(\hat{k})$ is the optimal solution of **(P1)**.*

Proof. Due to Proposition 2 and the fact that $\bar{\zeta}_{ik}+w_{ik}\geq 0$, it is sufficient to show that $\hat{A}^{opt}_{k,ii}>0$ for $i=2,...,N_k$ in **(P2)**, where

$$\hat{A}^{opt}_k(\cdot):=\sum_{\{i,k\}\in\mathcal{E},m=[4]}\Big(h^{opt}_{m,ki}B_{m,ki}\Big)+\hat{\lambda}^{opt}_kY_{k,r}+\hat{\gamma}^{opt}_k\bar{Y}_{k,r}.$$

We will first show that $\hat{\lambda}^{opt}_k\geq 0$ and $\hat{\gamma}^{opt}_k\geq 0$ for all $k\in\mathcal{N}$. By the complementary slackness condition, $R^{p,opt}_k$ satisfies

$$\mathrm{Tr}\{R^{p,opt}_k\begin{bmatrix}c_{k1}P^{opt}_k-a^{opt}_k & \star\\ \sqrt{c_{k2}}P^{opt}_k & -1\end{bmatrix}\}=0,\tag{8}$$

where $P^{opt}_k=\mathrm{Tr}\{Y_{k,r}W^{opt}_k\}+P_{D_k}$. By Proposition 3 and the minimization of L_0 of **(DP2)**, $r^{p,opt}_{k,22}=(r^{p,opt}_{k,12})^2$ for $R^{p,opt}_k$. Equation (8) becomes to

$$2r^{p,opt}_{k,12}\sqrt{c_{k2}}P^{opt}_k=(r^{p,opt}_{k,12})^2-c_{k1}P^{opt}_k+a^{opt}_k,$$

where a^{opt}_k should equal to $c_{k2}(P^{opt}_k)^2+c_{k1}P^{opt}_k$ for optimizing **(P2)**. Hence, $r^{p,opt}_{k,12}=\sqrt{c_{k2}}P^{opt}_k$ and $c_{k1}+2r^{p,opt}_{k,12}\sqrt{c_{k2}}\geq 0$ due to condition (b). Similar argument applies to $R^{q,opt}_k$ and we have $d_{k1}+2r^{q,opt}_{k,12}\sqrt{d_{k2}}\geq 0$. Using the conditions (a), we

have $\hat{\lambda}_k^{opt} \geq 0$ and $\hat{\mu}_k^{opt} \geq 0$ for all $k \in \mathcal{N} \setminus \mathcal{N}_I$. For $k \in \mathcal{N}_I$, we need to show

$$2r_{k,12}^{r,opt} + c_{k1} + 2\sqrt{c_{k2}}r_{k,12}^{p,opt} \geq 0,$$
$$2r_{k,13}^{r,opt} + d_{k1} + 2\sqrt{d_{k2}}r_{k,12}^{q,opt} \geq 0.$$

The complementary slackness condition suggests that

$$\mathrm{Tr}\{R_k^r \begin{bmatrix} -\bar{S}_k^2 & \star & \star \\ P_k^{opt} & -1 & \star \\ Q_k^{opt} & 0 & -1 \end{bmatrix}\} = 0.$$

If $(P_k^{opt})^2 + (Q_k^{opt})^2 = \bar{S}_k^2$, one can show that $r_{k,12}^{r,opt} = P_k^{opt}, r_{k,13}^{r,opt} = Q_k^{opt}$ by Proposition 3 and the complementary slackness condition above. If $(P_k^{opt})^2 + (Q_k^{opt})^2 < \bar{S}_k^2$, we can show $|r_{k,12}^{r,opt}| < P_k^{opt}$ and $|r_{k,13}^{r,opt}| < Q_k^{opt}$. In addition, $r_{k,12}^{r,opt} P_k^{opt} \geq 0$ and $r_{k,13}^{r,opt} Q_k^{opt} \geq 0$. By conditions (a) and (c), $\hat{\lambda}_k^{opt} \geq 0$ and $\hat{\mu}_k^{opt} \geq 0$ for $k \in \mathcal{N}_I$ follow.

Inspired by the condition $\hat{\lambda}_k^{opt} \geq 0$ and $\hat{\mu}_k^{opt} \geq 0$ for all $k \in \mathcal{N}$, we define a perturbed dual problem, **(δDP2)**, of **(DP2)** by changing the constraints on $\hat{\lambda}_k$ and $\hat{\mu}_k$ to $\hat{\lambda}_k \geq \delta > 0$ and $\hat{\mu}_k \geq \delta > 0$ for all k. By the strict feasibility of **(DP2)**, **(δDP2)** is strictly feasible with sufficient small δ, so the KKT conditions are the necessary and sufficient conditions for the optimal solution. If $h_{3,ki}^{\delta} = 0$, all $\hat{A}_k^{\delta}$ has $\mathrm{Re}(\hat{A}_{k,1i}^{\delta}) \neq 0$ because the power network is connected. In such case, $\mathrm{rank}(W_k^{\delta}) = 1$ because $\hat{A}_{k,ii}^{\delta} > 0$ for $i = 2, ..., N_k$ to ensure $\hat{A}_k^{\delta} \succeq 0$ for all $\forall k \in \mathcal{N}$. If $\exists h_{3,ki}^{\delta} \neq 0$ for an edge $\{i,k\} \in \mathcal{E}$ such that the non-zero real part of $\hat{A}_{k,1i}^{\delta}$ (and $\hat{A}_{k,i1}^{\delta}$) is cancelled out, $\mathrm{Re}(\hat{A}_{i,1k}^{\delta}) \neq 0$ as a counterpart. The rank of W_i^{δ} remains as one. Since the primal feasibility condition shown in Eq. (5) remains unchanged for **(δDP2)**, we have $\mathrm{rank}(W_k^{\delta}) = 1$ by having $\mathrm{rank}(W_i^{\delta}) = 1$. The argument can be extended to the case where there are multiple edges with non-zero $h_{3,ki}^{\delta}$, see Appendix. As argued above, the perturbed optimization **(δDP2)** has $\mathrm{rank}(W_k^{\delta}) = 1$ for all k.

Consider the bounded sequence of the primal-dual optimal solution of **(δDP2)**, $(y^{\delta}, h^{\delta}, R^{\delta}, W_{\mathcal{N}}^{\delta}, a^{\delta}, b^{\delta})$ with $\delta \to 0$, there is a converging subsequence where $\mathrm{rank}(W_k^{\delta})$ less than or equal to one. Since the set of positive semi-definite matrices with rank less than or equal to one is closed, the limit point of the converging sequence should also have the $\mathrm{rank}(W_k^{\delta}) = 1$, so $\mathrm{rank}(W_k^{opt}) = 1$ for all $k \in \mathcal{N}$. We complete the proof by Proposition 1. □

Theorem 1 is related to many existing OPF results. For example, Theorem 1 can contain Theorem 3 in [7] as a special case. Theorem 1 is also related to other existing conditions or observations for the exact SDP convex relaxation discussed in [13].

IV. Near Global Optimal Solution

Recognizing that Theorem 1 requires load over-satisfaction and certain conditions on the cost function, we seek an alternative approach when those conditions do not hold. It has been observed that SDP can provide a lower bound that is close to the global optimal value [21]. Slight modifications on the SDP approach may lead to a tight near global optimal solution. Notice that if the first condition in Proposition 3 is replaced by strict inequalities, then $\mathrm{rank}(A_k) \geq N_k - 1$ and the associated optimal W_k has rank one. We therefore consider the perturbed optimization

$$(\epsilon\textbf{DP2}) \qquad \max_{y \geq 0, h, R} L_0(y, h, R),$$

subject to

$$A_k(y, h, R) \succeq 0, \quad \forall k \in \mathcal{N},$$
$$A_{k,ii}(y, h, R) \geq \epsilon, \quad \forall i \in [N_k],$$
$$R_k^p \succeq 0, R_k^q \succeq 0, \quad \forall k \in \mathcal{N}_g \setminus \mathcal{N}_I,$$
$$R_k^r \succeq 0, \quad \forall k \in \mathcal{N}_I.$$

Clearly, the optimum of **(ϵDP2)** is no larger than the one of **(DP2)** and **(P2)** due to the smaller feasible region. If ϵ is chosen small enough, **(ϵDP2)** and **(DP2)** share close optima. To show how the solution of **(ϵDP2)** relates to **(P2)**, we consider the dual of **(ϵDP2)** given as follows

$$(\epsilon\textbf{DDP2}) \min_{a,b,W_k} \sum_{k \in \mathcal{N}_I} (a_k + b_k) + \sum_{\{i,k\} \in \mathcal{E}, \delta_{ik} \geq 0} (w_{ik}\mathrm{Tr}\{M_{ik}W_k\} - \epsilon\delta_{ik}),$$

subject to

$$\text{Eq. (4a-4f) hold}, \; W_k \succeq 0, \quad \mathrm{rank}(W_k) = 1, \; \forall k \in \mathcal{N},$$
$$\mathrm{Tr}\{M_{ik_r}W_k\} + \delta_{ik} \leq \bar{V}_{ik}, \; \forall \{i,k\} \in \mathcal{E},$$
$$B_{1,ki}W_k = B_{2,ik}W_i + \delta_{ik},$$
$$B_{l,ki}W_k = B_{l,ik}W_i, l = 3, 4.$$

The duality gap between **(ϵDDP2)** and **(ϵDP2)** is zero if **(DP2)** is strictly feasible and ϵ is sufficiently small. Notice that the solution of **(ϵDDP2)** may not satisfy constraint (5), keeping one from reconstructing a global optimum of **(P1)**. It can also be understood as the higher rank solutions are leveraged to the relaxation on constraint (5).

A. Near Global Optimal Algorithm for radial Networks

We propose a way to retrieve a optimum of **(P1)** if the network topology is radial and $\mathcal{N}_I = \emptyset$. Define a new optimization problem as follow

$$(\epsilon\textbf{RDP2}) \qquad \min_{W \succeq 0} \mathrm{Tr}\{AW\}, \quad \text{s.t. Eq. (3a-3g) hold},$$

where $A_{il} = \sum_{k \in \mathcal{N}} A_{k,\hat{i}\hat{l}}^{\epsilon}$, and A_k^{ϵ} is from the solution of **(ϵDP2)**. The inspiration of defining **(ϵRDP2)** is from the fact that the optimal solution pair of **(ϵDP2)** and **(ϵDDP2)** has $\mathrm{Tr}\{A_i^{\epsilon}W_i^{\epsilon}\} = 0$ for all $i \in \mathcal{N}$. Considering that W_i^{ϵ}, $\forall i \in \mathcal{N}$, is not necessary feasible to **(P1)** but close to its global optimum, **(ϵRDP2)** can help us to find a feasible solution of **(P1)** near W_i^{ϵ}, $\forall i \in \mathcal{N}$. The rank of A is either n or $n-1$ as shown in the following

1) $\exists \{i,k\} \in \mathcal{E}$ s. t. $\delta_{ik}^{\epsilon} > 0$ in **(ϵDDP2)**: $\mathrm{rank}(A) = N$ and $A \succ 0$.
2) $\delta_{ik}^{\epsilon} = 0$ for all $\{i,k\} \in \mathcal{E}$ in **(ϵDDP2)**: $\mathrm{rank}(A) = N - 1$ and $A \succeq 0$.

Remark 1. ***(Rank of A in Case (1))*** We briefly explain why $A \succ 0$ and $\mathrm{rank}(A) = N$ as follows. First, expend A_k^{ϵ} to a N dimensional Hermitian matrix by assigning zeros on the

new entries, which is denoted as $A^{\epsilon}_{N_k}$. In this notation, $A = \sum_{k\in\mathcal{N}} A^{\epsilon}_{N_k} \succeq 0$. It can be shown that $\cap_{k\in\mathcal{N}}\text{null}(A^{\epsilon}_{N_k}) = \emptyset$ if $\exists\{i,k\} \in \mathcal{E}$ s. t. $\delta^{\epsilon}_{ik} > 0$, which implies that all eigenvalues of A are strictly positive and $\text{rank}(A) = N$ follows.

Case (2) corresponds to the situation that the constraint $A_{k,ii}(\cdot) \geq 0, i \in [N_k]$, of **($\epsilon$DP2)** is inactive, which reduces **(ϵDP2)** to **(DP2)**. The optimal values of **(ϵDP2)** and **(P1)** are the same in this case. As a result, when $\text{rank}(A) = N-1$, the optimal value of **(ϵRDP2)** is 0 and its solution is also the optimum of **(P1)**. We can then only focus on the first case. When $A \succ 0$, **(ϵRDP2)** is the minimization of a strictly increasing function. In such case, the optimal solutions are on the Pareto front and geometric-based analyses apply. For the convenience of presenting the results of the geometric-based analysis, we introduce the equivalent form of constraint Eq. (4g) [13]

$$|\theta_i - \theta_k| \leq \Theta_{ik},\ \forall\{i,k\} \in \mathcal{E}, \tag{9}$$

where the voltage difference is translated into phase angle difference. The following theorem shows that **(ϵRDP2)** can provide a near global optimal solution of **(P1)** under a mild condition on the reactive power bounds

Theorem 2. *[9] If for all $k \in \mathcal{N}$,*

$$\underline{Q}_k < \sum_{\{i,k\}\in\mathcal{E}} Im\{\mathbf{Y}_{ik}\} - Im\{\mathbf{Y}_{ik}\}\cos(\Theta_{ik}) - Re\{\mathbf{Y}_{ik}\}\sin(\Theta_{ik}),$$

*then **(ϵRDP2)** with $\mathcal{N}_I = \emptyset$ has $\text{rank}(W^{opt}_{\epsilon}) = 1$.*

The proving idea of Theorem 2 is showing that the Pareto front of **(ϵRDP2)** is the same as the non-convex one with the constraint $\text{rank}(W) = 1$. The details of the proof can be found in Chapter 2 in [9]. By Theorem 2 and its mild assumption, a near global optimal solution of **(P1)** with convex quadratic objective function on the active and reactive power can always be found by solving **(ϵDP2)** and **(ϵRDP2)** in sequence. The error of the near global optimal solution is bounded by the difference of the optimal value between **(ϵRDP2)** and **(ϵDP2)**.

Remark 2. ***(Exact SDP convex relaxation on tree networks)*** Several existing works have shown that SDP (or SOCP) are exact convex relaxation on the OPF problems associated with tree networks [10], [22]. However, those results may not be applied to the general cases with quadratic reactive power cost functions added.

Remark 3. ***(Selection of ϵ)*** Although for any $\epsilon > 0$, we can argue that $\text{rank}(A) \geq N-1$, the numerical solution of **(ϵRDP2)** may not have a rank one optimum W^{ϵ} if ϵ is chosen too small. The second smallest eigenvalue of A^{ϵ}_i may be non-zero (but close to 0) for several $i \in \mathcal{N}$. In such case, $\text{rank}(A) \geq N-1$, but A has several eigenvalues close to zero. The numerical solution of **(ϵRDP2)** may have $\text{rank}(W^{\epsilon}) > 1$ as a consequence. We found that choosing $\epsilon \sim 5\%$ of $\underline{V}_k$ can avoid such numerical issue while preserving a small gap between **(ϵRDP2)** and **(ϵDP2)**.

B. Near Global Optimal Algorithm for Mesh Networks

For the mesh networks with $\mathcal{N}_I \neq \emptyset$, the geometric analyses become more challenging. No similar results as the radial networks are available. An alternative approach is proposed to retrieve a near global optimal solution with bounded violation on the constraint sets for mesh networks.

Recall that solving **(ϵDDP2)** gives the optimum with $\text{rank}(W^{\epsilon}_i) = 1\ \forall i \in \mathcal{N}$, or $W^{\epsilon}_i = (\hat{V}^{\epsilon}_i)^*\hat{V}^{\epsilon}_i$ equivalently. We propose to reconstruct the optimum of **(P1)** by defining a near global optimum, W^{opt}_{ϵ}, by Eq. (10)

$$W^{opt}_{\epsilon} = V^{opt}_{\epsilon}(V^{opt}_{\epsilon})^*, \quad V^{opt}_{\epsilon}(i) = \hat{V}^{\epsilon}_i(\hat{i}). \tag{10}$$

Let $\mathcal{E}_d$ be the collection of edges where $\delta_{ik} > 0$. The retrieved solution W^{opt}_{ϵ} and W^{ϵ}_i are compared as follows

$$\begin{cases} |W^{opt}_{\epsilon}(i,k)| > |W^{\epsilon}_i(\hat{i},\hat{k})|, & \text{for } \{i,k\} \in \mathcal{E}_d, \\ \angle W^{opt}(i,k) = \angle W^{\epsilon}_i(\hat{i},\hat{k}), & \text{for } \{i,k\} \in \mathcal{E}_d, \\ W^{opt}_{\epsilon}(i,i) > W^{\epsilon}_k(\hat{i},\hat{i}), & \text{for } \{i,k\} \in \mathcal{E}_d, \\ W^{opt}_{\epsilon}(i,k) = W^{\epsilon}_i(\hat{i},\hat{k}), & \text{otherwise.} \end{cases} \tag{11}$$

The discrepancy on the off-diagonal entries may result in a higher optimal value for **(P1)** compared to **(ϵDDP2)**. Several constraints that are related to those off-diagonal terms may also be violated. It is therefore necessary to analyze the bound of the errors.

Theorem 3. *If the optimum of **(P1)** is chosen by Eq. (10), then the following conditions hold*

1) *Only the constraints (3c-3e) may be violated for bus i with at least one k such that $\{i,k\} \in \mathcal{E}_d$. The errors are bounded by*

$$\begin{cases} \sqrt{\nu_i} & \text{for constraint (3c),} \\ \nu_i & \text{for constraint (3d) or (3e),} \end{cases} \tag{12}$$

where

$$\nu_i = \sum_{k:\{i,k\}\in\mathcal{E}_d} 2|Y_i(\hat{i},\hat{k})|\bar{V}^2_i\Big(\sqrt{1+\delta^{opt}_{ik}} - 1\Big)$$

2) *The difference of the optimal value between **(P1)** and **(ϵDDP2)** is bounded by*

$$\sum_{k\in\mathcal{N}_I} \nu_k\Big(|c_{k2}|(\bar{P}+\nu_k) + |d_{k2}|(\bar{Q}+\nu_k) + |c_{k1}| + |d_{k1}|\Big).$$

Proof. First of all, the constraint (3f) remains satisfied because $W^{opt}(i,i) = W^{\epsilon}_i(\hat{i},\hat{i})$ for all $i \in \mathcal{N}$. It can be shown that the voltage difference constraint (3g) remains satisfied by the direct computation as follows

$$\begin{aligned} &\text{Tr}\{M_{ik,r}W^{\epsilon}_i\} + \delta_{ik} \\ &= W^{\epsilon}_i(\hat{i},\hat{i}) + W^{\epsilon}_i(\hat{k},\hat{k}) - (W^{\epsilon}_i(\hat{i},\hat{k}) + W^{\epsilon}_i(\hat{k},\hat{i})) + \delta^{opt}_{ik} \\ &= W^{opt}_{\epsilon}(i,i) + W^{opt}_{\epsilon}(k,k) - (W^{\epsilon}_i(\hat{i},\hat{k}) + W^{\epsilon}_i(\hat{k},\hat{i})) \\ &> W^{opt}_{\epsilon}(i,i) + W^{opt}_{\epsilon}(k,k) - (W^{opt}_{\epsilon}(i,k) + W^{opt}_{\epsilon}(k,i)) \\ &= \text{Tr}\{M_{ik}W^{opt}_{\epsilon}\}. \end{aligned}$$

The second equality is by the constraint, $B_{1,ki}W_k = B_{2,ik}W_i + \delta_{ik}$, of **($\epsilon$DDP2)** and the strict inequality is a consequence of Eq. (11). Thus, we have

$$\text{Tr}\{M_{ik}W^{opt}_{\epsilon}\} < \text{Tr}\{M_{ik,r}W^{\epsilon}_i\} + \delta^{opt}_{ik} \leq \bar{V}_{ik}.$$

The discrepancy on the active power flows from bus i to k computed by W_ϵ^{opt} and W_i^ϵ is given as

$$Y_i(k,i)W_\epsilon^{opt}(i,k)+Y_i(i,k)W_\epsilon^{opt}(k,i) - \Big(Y_{i,r}(\hat{k},\hat{i})W_{\epsilon,i}^{opt}(\hat{i},\hat{k})+Y_{i,r}(\hat{i},\hat{k})W_{\epsilon,i}^{opt}(\hat{k},\hat{i})\Big)$$

As shown in Eq. (11), every entry of W_ϵ^{opt} preserves the phase angle of $W_k^\epsilon, k\in\mathcal{N}$. The equation above is then rewritten as

$$\Big(\sqrt{1+\delta_{ik}^{opt}}-1\Big)\Big(Y_{i,r}(\hat{k},\hat{i})W_i^\epsilon(\hat{i},\hat{k})+Y_{i,r}(\hat{i},\hat{k})W_i^\epsilon(\hat{k},\hat{i})\Big) \tag{13}$$
$$\le 2|Y_i(\hat{i},\hat{k})|\bar{V}_i^2\Big(\sqrt{1+\delta_{ik}^{opt}}-1\Big).$$

Equation (13) leads to the error bounds for the constraints (3d), (3e) and the optimal value. The bound for the error of constraint (3c) is derived by the similar argument. □

In practical OPF problems, $\bar{V}_{ik}$ is usually small with per unit representation on the voltage magnitude. As a consequence, $\sqrt{1+\delta_{ik}^{opt}}-1\le\sqrt{1+\bar{V}_{ik}}-1$ is usually small. In such case, solving **(ϵDDP2)** and retrieving the near global optimum of **(P1)** by Eq. (10) provides a near global optimal solution with small and bounded errors.

Remark 4. ***(The feasibility of the near global optimal solution).*** It may appear unreasonable to view the solution of Eq. (10) as a near global optimal solution as it might be infeasible for **(P1)**. However, the constraints that might be violated are the power equality/inequality constraints that are based on the prediction of loads or renewable generation. The predictions inherit a margin of estimation error. Hence, the solution with bounded violation on those constraints is practically acceptable for **(P1)**. In fact, similar concern can found in other penalization-based OPF solvers [13], [21], while we further provide bounds on the violation of the constraints.

Remark 5. ***(Distributed algorithms for solving the SDP relaxed OPF problem).*** In some scenarios of smart grid, distributed algorithms are desired because they allow near real-time implementation and each bus can have the cost function local and private. The proposed SDP-based algorithms (global or near global versions) are naturally distributed. The property opens the possibility of developing some distributed algorithms such as variances of primal-dual gradient method or alternating direction method of multipliers (ADMM) for the realization of a near real-time OPF solution.

Various optimizations are proposed to either find an exact or a near global optimum. We summarize the relations between the optimizations in Fig. 1. One can decide which optimization to solve depending on the applications.

V. Simulation Studies

We validate the proposed convex relaxation approach against New England 39 bus system and IEEE benchmark testbeds. The data of those examples are downloaded from [23]. The SDP convex problems are solved by cvx toolbox [24]. The objective function is defined in the form

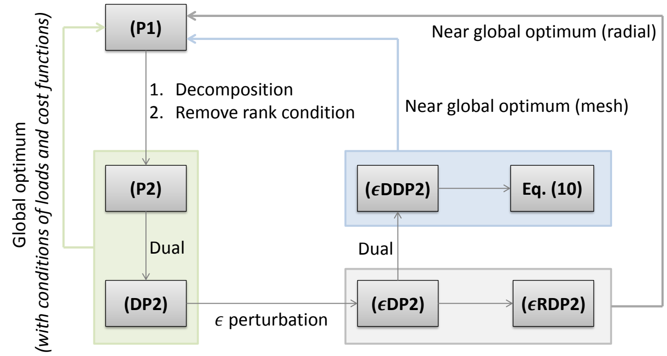

Fig. 1. Relations between the optimizations.

of Eq. (2). We first simulate the case when the conditions of Theorem 1 holds, the optimal solution of **(P2)**, $W_\mathcal{N}^{opt}$, has $\text{rank}(W_k^{opt})=1,\ \forall k\in\mathcal{N}$. As illustrated in Table I, load over-satisfaction condition almost leads no change on the final value of the optimization. Furthermore, only IEEE 57 bus and New England 39 bus system have one bus with the load over-satisfaction condition being active. Therefore, the load over-satisfaction condition required in Theorem 1 is conservative.

TABLE I
Simulation results of **(P2)**

	Opt. val.	Opt. val. w/o load over-sat.	#bus w. $\hat{\lambda}_k<0$ or $\hat{\gamma}_k<0$
IEEE 14	13915.2	13915.2	0
IEEE 30	7673.76	7673.76	0
IEEE 57	292843	292895	1
Eng. 39	421423	421423	1

All the test cases above has $\text{rank}(W_k^{opt})=1,\ \forall k\in\mathcal{N}$, in which case the near global optimal algorithm is unnecessary. In the following, we consider the objective functions which are tested in [8], [13] for IEEE 57 bus and New England 39 bus testbeds. Since we impose a looser constraint on the edge power flow compared to [8], [13], the solution of **(P2)** does not have $\text{rank}(W_k^{opt})=1,\ \forall k\in\mathcal{N}$. The buses with $\text{rank}(W_k^{opt})>1$ is shown in Table II. It is observed those buses have rank two and connect to at least a one purely inductive line. More specifically, $\{4,18\},\{9,55\},\{14,46\}\in\mathcal{E}$ in IEEE 57 bus testbed, and $\{2,30\}\in\mathcal{E}$ in New England 39 bus testbed are all purely inductive. Perturbing the admittance of the lines by adding small resistance may make the $\text{rank}(W_k^{opt})=1,\ \forall k\in\mathcal{N}$, see Corollary 2 in [7]. However, the assumptions of Corollary 2 in [7] are guaranteed to hold only if load over-satisfaction is allowed. We therefore resort to the proposed near global optimal algorithm in section IV-B. The perturbed optimization **(ϵDDP2)** is first solved to find $W_{\epsilon,i}^{opt}, i\in\mathcal{N}$, where $\text{rank}(W_{\epsilon,i}^{opt})=1$ and ϵ is chosen as 0.02. The near global optimal solution W_ϵ^{opt} is then computed from Eq. (10). The derived near global optimum W_ϵ^{opt} satisfies all the constraints in the generation buses and line power limits. The violations on loads equality constraints are less than $10^{-4}(p.u.)$ in both IEEE 57 and New England 39 testbeds, which is acceptable as discussed in Remark 4. The optimal values are summarized in Table III.

TABLE II
BUSES THAT HAS RANK(W_k^{opt}) > 1 FOR **(P2)**

	IEEE 57	Eng. 39
Bus	4 9 14 18 46 55	2 30
Edge	$\{4,18\},\{9,55\},\{14,46\}$	$\{2,30\}$

TABLE III
NEAR GLOBAL OPTIMAL SOLUTION ($\epsilon = 0.02$)

	IEEE 57	Eng. 39
Opt. val. of (ϵ**DDP2**)	41712.8	41852.6
Opt. val. for W_ϵ^{opt} of Eq. (10)	41712.9	41852.7
% error	2.39×10^{-8}	2.4×10^{-8}

VI. CONCLUSION

In this paper, we propose distributed SDP methods to solve the OPF problems with reactive power cost functions and apparent power constraints. The novel convex relaxation on the distributed formulation of the OPF problem is proposed to derive new conditions of exact SDP convex relaxation. More importantly, we develop algorithms to find a near global optimum for the OPF problems such that those conditions are not held. We further show that the error of the near global optimum is bounded by the angle difference between connected buses. Simulation results demonstrate that the errors are small. Validating the algorithms on more test cases is among our future works.

APPENDIX

Supplementary Proof of Theorem 1 In the proof of Theorem 1, we only consider the case with only one edge having $h_{3,ki}^{opt} = -h_{3,ki}^{opt} \neq 0$. If there are multiple $\{l,m\} \in \mathcal{E}$ such that $h_{3,lm}^{opt} \neq 0$, then it is possible that two connected buses i,k have rank$(\hat{A}_i^\delta) < N_i - 1$ and rank$(\hat{A}_k^\delta) < N_k - 1$. The arguments in the proof of Theorem 1 may no longer hold. Fortunately, we can show that rank(W_k^δ) =rank$(W_i^\delta) = 1$ even if the rank of $\hat{A}_i^\delta$ and $\hat{A}_k^\delta$ are reduced by some $h_{3,lm}^{opt} \neq 0, \{l,m\} \in \mathcal{E}$.

It is sufficient to show that for bus k, there exists a unique unitary $V_k^\delta \in \mathbb{C}^{N_k}$ such that the following equalities hold.

$$B_{1,ki} V_k^\delta (V_k^\delta)^* = B_{2,ik} W_i^\delta, \quad \{i,k\} \in \mathcal{E} \tag{14a}$$

$$B_{l,ki} V_k^\delta (V_k^\delta)^* = B_{l,ik} W_i^\delta, l = 3,4, \tag{14b}$$

$$\mathrm{Tr}\{A_k^\delta V_k^\delta (V_k^\delta)^*\} = 0 \tag{14c}$$

Define $\mathcal{B}_k := \{i | h_{3,ki}^\delta = 0, i \in \mathcal{V}_k\}$. There exists a unique unitary $V_{\mathcal{B}_k}^\delta \in \mathbb{C}^{|\mathcal{B}_k|}$ such that

$$\mathrm{Tr}\{A_k^\delta \begin{bmatrix} V_{\mathcal{B}_k}^\delta \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} (V_{\mathcal{B}_k}^\delta)^* & \mathbf{0} \end{bmatrix}\} = 0,$$

where without loss of generality, we label the first $|\mathcal{B}_k|$ coordinates such that they span the range of $\hat{A}_k^\delta$. For $i \in \mathcal{V}_k \backslash \mathcal{B}_k$, W_i^δ is not necessary to have rank one because $h_{3,il}^\delta \neq 0$ might be the case for some $\{i,l\} \in \mathcal{E}$. However, the following property still holds for W_i^δ:

$$\mathrm{rank}\Big(\begin{bmatrix} W_i^\delta(\hat{i},\hat{i}) & \star \\ W_i^\delta(\hat{k},\hat{i}) & W_i^\delta(\hat{k},\hat{k}) \end{bmatrix} \Big) = 1.$$

Accordingly, there exists unique $V_k^\delta(\hat{i}) \in \mathbb{C}$ for $i \in \mathcal{V}_k \setminus \mathcal{B}_k$ such that Eq. (14a) and (14b) hold, which completes the proof.

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